2
Quantum Electrical Circuits

2.1 Introduction

Quantum electrodynamics is the theory of interaction between electrons (and atoms) with electromagnetic fields. These lecture notes discuss the closely related problem of quantization of electrical circuits (Devoret, 1997; Schoelkopf and Girvin, 2008). Experimental progress over the last decade in creating and controlling quantum coherence in superconducting electrical circuits has been truly remarkable. The quantum electrodynamics of superconducting microwave circuits has been dubbed 'circuit QED' by analogy to cavity QED in quantum optics. These lecture notes will describe the quantum optics approach to microwave circuits with superconducting qubits playing the role of artificial atoms whose properties can be engineered. Despite being large enough to be visible to the naked eye, these artificial atoms have a very simple discrete set of quantized energy levels which are nearly as well understood (Nigg *et al.*, 2012) as those of the prototypical single-electron atom, hydrogen. Furthermore it has proven possible to put these atoms into coherent superpositions of different quantum states so that they can act as quantum bits. Through clever engineering, the coherence times of such superposition states has risen more than four orders of magnitude from nanoseconds for the first superconducting qubit created in 1999 (Nakamura *et al.*, 1999) up to $\sim 30 - 150$ microseconds today (Paik *et al.*, 2011; Rigetti *et al.*, 2012; Chang *et al.*, 2013; Barends *et al.*,). Recent experiments with the fluxonium qubit design (Manucharyan *et al.*, 2009*b*) have achieved T_1 times exceeding 1 millisecond (Geerlings *et al.*, 2013). 'Schoelkopf's Law' for the exponential growth of coherence time is illustrated in Fig. (2.1).

Simple quantum machines have already been built using superconducting circuits which can manipulate and measure the states of individual qubits (Nakamura *et al.*, 1999; Mooij *et al.*, 1999; Vion *et al.*, 2002) as well as individual microwave quanta (Houck *et al.*, 2007; Hofheinz *et al.*, 2008; Hofheinz *et al.*, 2009; Johnson *et al.*, 2010; Mariantoni *et al.*, 2011*a*; Wang *et al.*, 2011), entangle two (Ansmann *et al.*, 2009; Chow *et al.*, 2010) and three qubits (Neeley *et al.*, 2010; DiCarlo *et al.*, 2010), run simple quantum algorithms (DiCarlo *et al.*, 2009; Mariantoni *et al.*, 2011*b*) and perform rudimentary quantum error correction (Reed *et al.*, 2012). Future improved qubit designs, microwave circuit designs, and materials improvements should allow this trend to continue unabated. In addition to being a potentially powerful engineering architecture for building a quantum computer, circuit QED opens up for us a novel new regime to study ultra-strong coupling between 'atoms' and individual microwave photons (Devoret *et al.*, 2007). The concept of the photon is a subtle one, but hopefully these notes

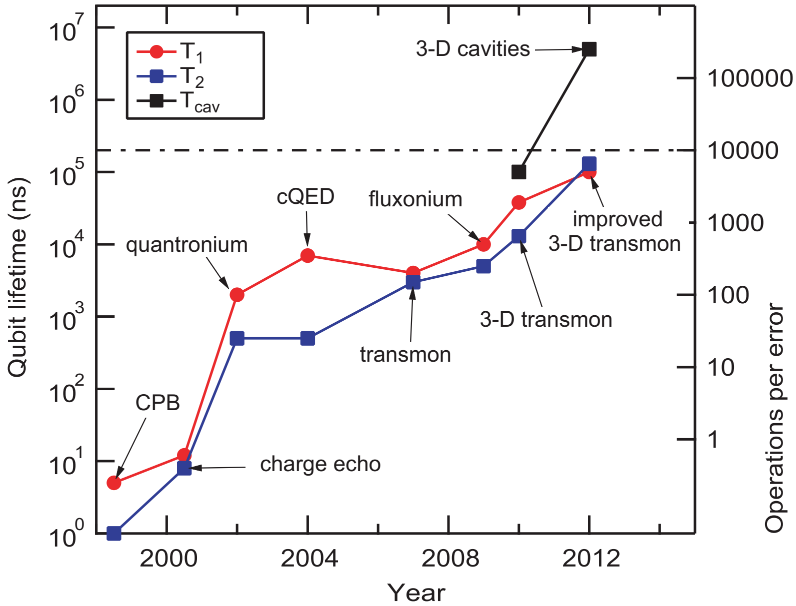

Fig. 2.1 "Schoelkopf's Law" plot illustrating the exponential growth for superconducting (charge-) qubit coherence times. Recent experiments (Geerlings *et al.*, 2013) with the 'fluxonium' qubit design have achieved T_1 times exceeding one millisecond.

will convince the reader that microwaves, despite their name, really are particles. We will accordingly begin our study with a review of the quantization of electromagnetic fields in circuits and cavities.

The quantization of electrical circuits has been thoroughly addressed in the Les Houches lecture notes of my colleague, Michel Devoret (Devoret, 1997), to which I direct the interested reader. The circuit elements that are available to the quantum engineer include those familiar to classical engineers: resistors, capacitors, and inductors. Resistors cause unwanted dissipation and we will attempt to avoid them. See however further discussion in the Appendix (B) of spontaneous emission into transmission lines which act effectively as fixed impedances. Dissipation into a cold resistor can in fact be useful for qubit reset(Reed *et al.*, 2010*b*) to the ground state since reset requires removal of entropy to a cold bath.

In addition to these standard circuit elements, there is one special element in superconducting circuits, the Josephson tunnel junction. We will be learning more about superconductivity and Josephson junctions later, but for now we simply note the following. With capacitors and inductors we can build simple LC harmonic oscillators. If we can eliminate all resistors then the harmonic oscillations will be undamped. The use of superconducting circuits takes us a long way towards this goal of zero dissipation, about which more later. The essential feature of (ordinary) superconductivity is that electrons of opposite spin pair up and condense into a special ground state with a substantial excitation gap 2Δ needed to break one of the pairs and create an excited state. This pair excitation gap is essential to the ability of current to flow in a superconductor without dissipation.[1] A closely related advantage of the excitation gap is that it dramatically reduces the number of effective degrees of freedom in the circuit, allowing us to construct artificial 'atoms' that behave like simple single-electron atoms even though they are made up of $10^9 - 10^{12}$ aluminum atoms. The extremely powerful force of the Coulomb interactions also plays an essential role in limiting the low energy degrees of freedom in circuits. When the Coulomb interaction is unscreened, the gapless collective motion of currents is lifted up to the plasma frequency which is orders of magnitude higher than any relevant frequency scale for the circuits we will consider. (This effect of the long-range Coulomb force occurs in both normal metals and superconductors.) In the presence of screening due to ground planes or shields, the plasma oscillations are 'acoustic modes' with a linear dispersion and velocity close to the speed of light in vacuum.[2] When quantized, these will be our propagating photons.

2.2 Plasma Oscillations

Because the powerful effect of long-range Coulomb interactions plays a crucial role in simplifying the spectrum of quantum electrical circuits, let us begin our analysis by reviewing the plasma oscillations in a bulk metal. Throughout this work we will use SI units. We will consider infinitesimal density fluctuations δn around the mean electron number density n. In the 'jellium' model the mean charge density is canceled by the ionic background so the net charge density is

$$\rho(\vec{r}) = -e\,\delta n. \tag{2.1}$$

The current flowing (to zeroth order in δn) is

$$\vec{J}(\vec{r}, t) = -en\vec{v}(\vec{r}, t), \tag{2.2}$$

where the local electron mean velocity field obeys Newton's law

$$\frac{\partial}{\partial t}\vec{v} = \frac{-e}{m}\vec{E}, \tag{2.3}$$

where m is the electron (effective) mass. This in turn yields

$$\frac{\partial}{\partial t}\vec{J} = \frac{ne^2}{m}\vec{E}. \tag{2.4}$$

Taking the divergence of both sides of this equation and applying Gauss's law

[1] There do exist gapless superconductors (e.g. d-wave materials like YBCO) which can carry a dc current without dissipation, but at the microwave frequencies of interest for qubits, the lack of a gap implies significant dissipation.

[2] Flat metallic surfaces and long wires exhibit so-called surface plasmons which are gapless and have approximately linear dispersion relations due to electrodynamic retardation effects. The purpose of the ground shield surrounding the central wire in a coaxial cable is to prevent radiation losses when the cable is bent into a curve.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \tag{2.5}$$

and the continuity equation

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial}{\partial t}\rho = 0, \tag{2.6}$$

yields

$$\frac{\partial^2}{\partial t^2}\rho = -\omega_{\mathrm{p}}^2 \rho \tag{2.7}$$

where the so-called ‘plasma frequency’ is given by[3]

$$\omega_{\mathrm{p}}^2 \equiv \frac{ne^2}{m\epsilon_0}. \tag{2.8}$$

Electromagnetic waves cannot propagate in a plasma at frequencies below the plasma frequency (Jackson, 1999). In the earth’s ionosphere, the typical plasma frequency is in the range of 10’s of MHz and varies between night and day, thereby affecting short-wave radio reception. In the typical metals we will be concerned with (e.g., aluminum), the valence electron density is sufficiently high that the plasma frequency is in the ultraviolet region of the optical spectrum. Hence aluminum (whose plasma frequency $\omega_{\mathrm{p}}/(2\pi) \sim 3.6 \times 10^{15}$Hz corresponds to a photon energy of $\sim$ 15 eV) is highly reflective in the visible. Essentially, the electrons are so dense and so agile that they screen out any electric fields almost perfectly over a very short screening distance. For frequencies far below the plasma frequency, Maxwell’s equations yield

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} \approx -\lambda_{\mathrm{p}}^{-2}\vec{E}, \tag{2.9}$$

where the London penetration depth, λ_{L}, of the electromagnetic fields is

$$\lambda_{\mathrm{L}} = \frac{c}{\omega_{\mathrm{p}}} = \frac{1}{\sqrt{4\pi n r_{\mathrm{e}}}}, \tag{2.10}$$

where the classical radius of the electron is given by

$$r_{\mathrm{e}} = \frac{e^2}{4\pi\epsilon_0}\frac{1}{mc^2} \approx 2.818 \times 10^{-15}\mathrm{m}. \tag{2.11}$$

For Al, Eq. 2.10 yields[4] $\lambda_{\mathrm{L}} \sim$ 14nm We will be dealing with GHz frequency scales many orders of magnitude below the plasma frequency and centimeter wavelength scales relative to which the penetration depth is effectively zero.

[3] We neglect here the various details of the band structure of Al as well as the possibility that the core electrons in the atoms of the metal contribute a dielectric constant $\epsilon \neq 1$ seen by the valence electrons whose dynamics create the plasma oscillations of the metal.

[4] The measured value of the London penetration depth in Al (at zero frequency) is somewhat larger, $\lambda_{\mathrm{L}} \sim$ 51.5nm. The difference is presumably due to variation in the core electron dielectric constant with frequency which has been neglected in our model. It should also be noted that in dirty superconductors, the reduction in the superfluid stiffness causes the penetration depth to increase.

Exercise 2.1 Derive Eq. 2.9 in the limit of low frequencies and show that it leads to exponential decay of transverse electromagnetic waves with decay length λ_{p}.

The above simplified[5] jellium model yields a plasma mode which is completely dispersionless–the mode frequency is independent of wave vector q. The frequency of the bulk collective plasma mode is vastly higher than any microwave frequency that we will be dealing with. From the point of view of quantum mechanics, the amount of energy required to create a bulk plasmon is so large that we can consider these degrees of freedom to be frozen into their quantum mechanical ground state. Hence they can be ignored. The approximations leading to Eq. (2.2) breakdown at short distances due to the granularity of the electron charge. At very large wave vectors approaching the Fermi wave vector, the jellium continuous charge picture breaks down and the plasma oscillation frequency rises and the mode becomes 'Landau-damped' due to the collective charge oscillation mode decaying into single-particle excitations (Pines, 1963). Conversely for extremely small wave vectors, there is a cutoff associated with the finite size of any sample. This we can take into account by considering the capacitance matrix between different lumps of metal in the circuit we are trying to quantize. In certain circumstances, the capacitance matrix is such that there do exist collective charge oscillation modes which are down in the microwave range. These will be the important modes which we will quantize. Here the superconductivity is vital for gapping the single-particle excitations so that the collective charge modes are both simple and extremely weakly damped.

2.3 Quantum LC Oscillator

The circuit element with the simplest dynamics is the LC oscillator illustrated schematically in Fig. (2.2a). Now that we understand that supercurrents can flow essentially without dissipation and that the great strength of the Coulomb interaction lifts density fluctuations up to optical frequencies, we can understand that the LC oscillator has, to a very good approximation, only a single low-energy degree of freedom, namely uniform divergenceless current flow in the wire of the inductor which does not build up charge anywhere except on the plates of the capacitor. This is a very good approximation in the 'lumped element' limit where the physical size of the LC oscillator is much smaller than than the wavelength of electromagnetic waves at the frequency of the oscillator, $\lambda = 2\pi c/\Omega$. [This caveat is associated with the unstated assumption in our discussion of plasma oscillations that we neglected electrodynamic retardation effects. That is, we effectively assumed $c = \infty$.] In terms of the capacitor charge q and the inductor current I the Lagrangian is readily written

$$\mathcal{L} = \frac{1}{2}LI^2 - \frac{1}{2}\frac{q^2}{C}. \tag{2.12}$$

Using charge conservation, $I = +\dot{q}$, this can be cast into the more familiar form

[5]A more careful treatment would have included the change in the Fermi energy as the density oscillates. The resulting Fermi pressure gradients produce a positive quadratic dispersion of the plasma mode with increasing wave vector.

$$\mathcal{L} = \frac{L}{2}\dot{q}^2 - \frac{1}{2C}q^2. \tag{2.13}$$

Remarkably, we have reduced a complex circuit containing an enormous number of electrons to a system with a *single* degree of freedom q with 'mass' L and 'spring constant' $1/C$). This is possible only because all but this one degree of freedom are effectively gapped out by a combination of superconductivity (which gaps out the single-particle excitations) and the long-range Coulomb force (which gaps out the collective plasmon (density fluctuation) degrees of freedom). All that is left is the rigid collective motion of the incompressible electron fluid sloshing back and forth, charging and discharging the capacitor.

Eq. (2.13) yields the Euler-Lagrange equation of motion

$$\ddot{q} = -\Omega^2 q, \tag{2.14}$$

where the natural oscillation frequency is

$$\Omega = \frac{1}{\sqrt{LC}}. \tag{2.15}$$

The momentum conjugate to the charge is the flux through the inductor

$$\Phi = \frac{\delta\mathcal{L}}{\delta\dot{q}} = L\dot{q} = LI. \tag{2.16}$$

Thus the Hamiltonian can be written

$$H = \Phi\dot{q} - \mathcal{L} = \frac{\Phi^2}{2L} + \frac{1}{2C}q^2. \tag{2.17}$$

Hamilton's equations of motion then give the current through the inductor and the voltage at the node connecting the inductor and the capacitor

$$\dot{q} = \frac{\partial H}{\partial \Phi} = \frac{\Phi}{L} = I \tag{2.18}$$

$$\dot{\Phi} = -\frac{\partial H}{\partial q} = -\frac{q}{C} = V. \tag{2.19}$$

In the usual way, the coordinate and its conjugate momentum can be promoted to quantum operators obeying the canonical commutation relation

$$[\hat{\Phi}, \hat{q}] = -i\hbar \tag{2.20}$$

and we can write the Hamiltonian

$$H = \frac{\hbar\Omega}{2}\left\{\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right\} = \hbar\Omega\left\{\hat{a}^\dagger\hat{a} + \frac{1}{2}\right\}, \tag{2.21}$$

in terms of raising and lowering operators

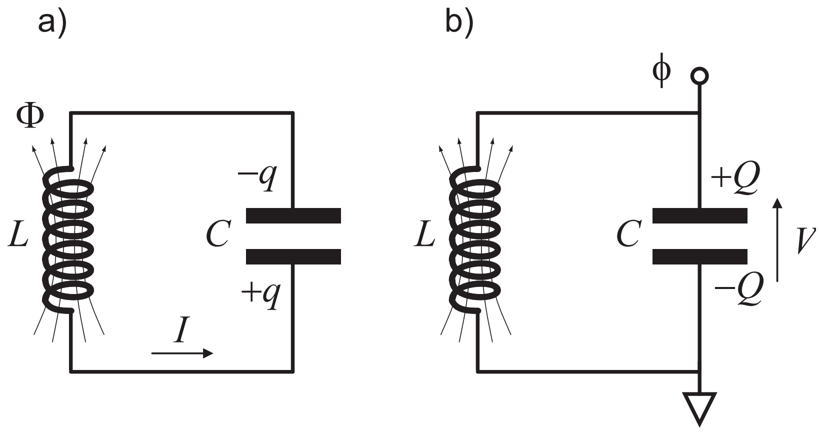

Fig. 2.2 Simple LC electrical oscillator analogous to a mass and spring mechanical oscillator. In panel a) the position coordinate of the mass is taken to be q, the charge accumulated on the capacitor by the current I flowing through the inductor, and the flux Φ through the inductor is the momentum. The sign convention for the charge is such that $\dot{q} = I$ and therefore the inductance L is analogous to the mass. The role of the spring constant is played by $1/C$ and the potential energy of the capacitor is $(q - q_q)^2/2C$, where q_0 is the offset charge of the capacitor (the equivalent of the equilibrium length of the spring). Hamilton's equation for the time rate of change of the momentum is $\dot{\Phi} = -(q - q_0)/C$. In panel b) the position coordinate is now taken to be ϕ, the time integral of the voltage V across the capacitor (i.e., the node flux) and the conjugate momentum is Q, the charge on the capacitor resulting from the electrochemical potential difference between the two plates. The role of the mass is played by C and the spring constant is now $1/L$, with the energy of the inductor given by $(\phi - \phi_0)^2/2L$, where ϕ_0 is the external flux in the loop of the circuit (including the coil of the inductor). Hamilton's equation for the time rate of change of position is $\dot{\phi} = Q/C$. Note the important sign change in the denition of charge: $Q = q_0 - q$, needed to make the Hamilton equations of motion correct in each case. The classical Poisson brackets and the quantum canonical commutation relations between position and momentum are maintained between the two cases: $[\hat{q}, \hat{\Phi}] = [\hat{\phi}, Q] = +i\hbar$.

$$\hat{a} = +i\frac{1}{\sqrt{2L\hbar\Omega}}\hat{\Phi} + \frac{1}{\sqrt{2C\hbar\Omega}}\hat{q} \tag{2.22}$$

$$\hat{a}^\dagger = -i\frac{1}{\sqrt{2L\hbar\Omega}}\hat{\Phi} + \frac{1}{\sqrt{2C\hbar\Omega}}\hat{q} \tag{2.23}$$

which obey the usual relation

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{2.24}$$

In the above discussion we chose the charge q on the capacitor as the natural coordinate of the harmonic oscillator and found that the inductor flux Φ was the momentum conjugate to this flux. In the picture we interpret the capacitance C as the inverse of the 'spring constant,' and the inductance L as the 'mass.' This seems natural from our intuitive view of the capacitance as storing the potential energy and the inductor storing the kinetic energy (actually the kinetic energy of the electrons makes only a small contribution (called the 'kinetic inductance') to the total inductance. It is primarily the energy stored in the magnetic field created by the current which dominates the inductance in most situations.)

When dealing with Josephson junctions we will start with this same representation but then find that they act as non-linear inductors and so it will be more convenient to take the node flux (defined below) to be the coordinate rather than the momentum. In order to get used to this alternative representation, we will practice here on the LC oscillator. Following Devoret (Devoret, 1997) let us define the node flux at the point shown in Fig. (2.2b) by

$$\phi(t) = \int^{t} d\tau V(\tau), \tag{2.25}$$

so that $V(t) = \dot{\phi}$. Then the potential energy stored on the capacitor is

$$U = \frac{1}{2}C\dot{\phi}^2 \tag{2.26}$$

and now looks like the kinetic energy with this choice of coordinate. Similarly, using Faraday's law and the sign convention for the direction of the current defined in Fig. (2.2b) we have

$$V = L\dot{I} = \dot{\phi} \tag{2.27}$$

and thus see that the node flux variable ϕ really is the physical magnetic flux Φ winding through the inductor (ignoring any possible external flux applied through the loop of the circuit or the inductor). Hence the kinetic energy stored in the inductor is

$$T = \frac{1}{2L}\phi^2, \tag{2.28}$$

which now looks like the potential energy. With this choice of coordinate the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}C\dot{\phi}^2 - \frac{1}{2L}\phi^2, \tag{2.29}$$

and the momentum conjugate to the flux

$$Q = \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = C\dot{\phi} \tag{2.30}$$

is now the charge as defined with the sign convention in Fig. (2.2b). Notice the crucial minus sign relative to the previous result. This is necessary to maintain the sign of the commutation relation when we interchange the momentum and coordinate. To reiterate: when the charge is the coordinate and the flux is the conjugate momentum, the commutation relation is:

$$[\hat{q}, \hat{\Phi}] = +i\hbar, \tag{2.31}$$

whereas when the flux is the coordinate and the charge is the conjugate momentum, the commutation relation is:

$$[\hat{\phi}, \hat{Q}] = +i\hbar. \tag{2.32}$$

Since we have chosen a convention in which $\hat{\Phi} = \hat{\phi}$, we require $\hat{Q} = -\hat{q}$.

Just to be completely explicit, we now repeat the derivation of the Hamiltonian and its quantization for this new choice which we will be using throughout the remainder of these notes. Thus the Hamiltonian can be written

$$H = Q\dot{\phi} - \mathcal{L} = \frac{1}{2C}Q^2 + \frac{\phi^2}{2L}. \tag{2.33}$$

Hamilton's equations of motion are then

$$\dot{\phi} = +\frac{\partial H}{\partial Q} = +\frac{Q}{C} \tag{2.34}$$

$$\dot{Q} = -\frac{\partial H}{\partial \phi} = -\frac{\phi}{L}. \tag{2.35}$$

Again in the usual way, the coordinate and its conjugate momentum can be promoted to quantum operators obeying the canonical commutation relation (but note the important position reversal from Eq. (2.20))

$$[\hat{Q}, \hat{\phi}] = -i\hbar \tag{2.36}$$

and we can write the Hamiltonian

$$H = \frac{\hbar\Omega}{2}\left\{\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right\} = \hbar\Omega\left\{\hat{a}^\dagger\hat{a} + \frac{1}{2}\right\}, \tag{2.37}$$

in terms of raising and lowering operators

$$\hat{a} = +i\frac{1}{\sqrt{2C\hbar\Omega}}\hat{Q} + \frac{1}{\sqrt{2L\hbar\Omega}}\hat{\phi} \tag{2.38}$$

$$\hat{a}^\dagger = -i\frac{1}{\sqrt{2C\hbar\Omega}}\hat{Q} + \frac{1}{\sqrt{2L\hbar\Omega}}\hat{\phi} \tag{2.39}$$

which obey the usual relation

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{2.40}$$

The charge and flux operators can be expressed in terms of the raising and lowering operators as

$$\hat{Q} = -iQ_{\mathrm{ZPF}}\left(\hat{a} - \hat{a}^\dagger\right) \tag{2.41}$$

$$\hat{\phi} = \Phi_{\mathrm{ZPF}}\left(\hat{a} + \hat{a}^\dagger\right), \tag{2.42}$$

where

$$Q_{\mathrm{ZPF}} = \sqrt{\frac{C\hbar\Omega}{2}} = \sqrt{\frac{\hbar}{2Z}} \tag{2.43}$$

$$\Phi_{\mathrm{ZPF}} = \sqrt{\frac{L\hbar\Omega}{2}} = \sqrt{\frac{\hbar Z}{2}}, \tag{2.44}$$

where Z is the characteristic impedance of the oscillator

$$Z = \sqrt{\frac{L}{C}}. \tag{2.45}$$

Notice that the notation has been chosen such that the quantum ground state uncertainties in charge and flux are given by

$$\langle 0|\hat{Q}^2|0\rangle = Q_{\mathrm{ZPF}}^2 \tag{2.46}$$
$$\langle 0|\hat{\phi}^2|0\rangle = \Phi_{\mathrm{ZPF}}^2. \tag{2.47}$$

Exercise 2.2 There is a certain arbitrariness in the choice of phase factors that enter in definition of the raising and lowering operators in Eq. (2.42). We have chosen a convention in which the flux is related to the real part of $\hat{a}$ and the charge is related to the imaginary part of $\hat{a}$. Consider the unitary transformation $U = e^{i\theta\hat{n}}$, where $\hat{n} = a^\dagger a$ is the photon number operator. What does this transformation do to the Fock state $|n\rangle$? How do the raising and lowering operators transform under the action of U? What happens to the expressions for charge and flux under the transformation of U when $\theta = \pi/2$?

Using the superconducting resistance quantum

$$R_{\mathrm{Q}} \equiv \frac{h}{(2e)^2} \approx 6,453.20\,\mathrm{Ohms}, \tag{2.48}$$

we can define a dimensionless characteristic impedance

$$z \equiv Z/R_{\mathrm{Q}}, \tag{2.49}$$

to obtain

$$Q_{\mathrm{ZPF}} = (2e)\sqrt{\frac{1}{4\pi z}} \tag{2.50a}$$

$$\Phi_{\mathrm{ZPF}} = \Phi_0\sqrt{\frac{z}{4\pi}}, \tag{2.50b}$$

where

$$\Phi_0 \equiv \frac{h}{2e} \tag{2.51}$$

is the superconducting flux quantum. Notice that the usual uncertainty product is obeyed.

$$Q_{\mathrm{ZPF}}\Phi_{\mathrm{ZPF}} = \frac{\hbar}{2}. \tag{2.52}$$

The voltage is an important physical variable and the voltage operator is given by

$$\begin{aligned}\hat{V} &= \frac{d\hat{\phi}}{dt} = \frac{i}{\hbar}[H, \hat{\phi}] \\ &= \frac{1}{C}\hat{Q} = -i\sqrt{\frac{\hbar\Omega}{2C}}\left(\hat{a} - \hat{a}^\dagger\right) = -iV_{\mathrm{ZPF}}\left(\hat{a} - \hat{a}^\dagger\right),\end{aligned} \tag{2.53}$$

where

$$V_{\mathrm{ZPF}} = \Omega\Phi_{\mathrm{ZPF}} = \Omega\Phi_0\sqrt{\frac{z}{4\pi}}. \tag{2.54}$$

The superconducting flux quantum in convenient units is given by

$$\Phi_0 \approx 2.06783367\,\mu\mathrm{V/GHz} \tag{2.55}$$

which tells us that the vacuum fluctuations of the voltage across the capacitor in a typical 10 GHz, $Z = 100$ Ohm impedance resonator circuit will be on the scale of $\sim (1/3)\mu$V. Correspondingly the vacuum fluctuations of the current are on the scale of ~ 3nA. It is remarkable that the quantum fluctuations of currents and voltages in these microwave circuits have the same scales as are routinely measured in the audio range with standard laboratory instruments.

How do we interpret the excitation quanta of this harmonic oscillator? We can think of these as excitations of the collective motion of the electrons in the wire, or we can think of them as photons of the electromagnetic field. Because this is a lumped element resonator (as opposed to a cavity or other distributed resonator), the electric field appears between the capacitor plates and the magnetic field appears in a separate place, namely within the coil of the inductor. Nevertheless it is perfectly acceptable to think of these excitations as photons. The coordinate of the oscillator is the flux in the coil (or in the first choice we made, the charge on the capacitor plates which is equivalent to the electric field in the gap between the plates.

One does not normally think about photons in the context of first quantization, but this is also useful for building up intuition and for thinking about things like the full probability distribution of electric field measurement results. The wave function of the vacuum state is a gaussian in the coordinate ϕ as shown in Fig. (2.3)

$$\Psi_0(\phi) = \frac{1}{[2\pi\Phi_{\mathrm{ZPF}}^2]^{1/4}} e^{-\frac{1}{4}\frac{\phi^2}{\Phi_{\mathrm{ZPF}}^2}}. \tag{2.56}$$

If in the vacuum state we make a precise measurement of the flux, the resulting value will be random and have a gaussian probability distribution given by

$$P(\phi) = |\Psi_0(\phi)|^2. \tag{2.57}$$

Hence the most probable value of the flux is zero. On the other hand, in the one-photon state

$$\Psi_1(\phi) = \frac{\phi}{\Phi_{\mathrm{ZPF}}} \frac{1}{[2\pi\Phi_{\mathrm{ZPF}}^2]^{1/4}} e^{-\frac{1}{4}\frac{\phi^2}{\Phi_{\mathrm{ZPF}}^2}} \tag{2.58}$$

zero flux would never be measured because the wave function vanishes at $\phi = 0$. The measured flux is still zero on average. This is true for any (odd) photon Fock state (number eigenstate) from simple parity considerations. On the other hand, if the photon number is uncertain, for example in the coherent superposition state

$$\Psi_+ = \frac{1}{\sqrt{2}}\left(\Psi_0 + \Psi_1\right), \tag{2.59}$$

then the centroid of the probability distribution is displaced away from zero as shown in Fig. (2.3) and the average value of the flux will be non-zero. A similar conclusion is readily reached within the second quantized formulation of Eq. (2.42) by noticing that the flux and charge operators are purely off-diagonal in the photon number basis.

Fig. 2.3 LC oscillator wave function amplitude (left panel) and probability density (right panel) plotted vs. the coordinate ϕ. Solid: ground state, Ψ_0; Long-Dashed: first excited state, Ψ_1; Short-dashed: linear combination of the ground and first excited states, $\frac{1}{\sqrt{2}}(\Psi_0 + \psi_1)$.

Such superpositions of zero and one-photon states cannot be achieved by simply weakly driving the oscillator as this produces a coherent superposition of all photon number states (to be described further below). However they have been achieved experimentally (Houck *et al.*, 2007; Hofheinz *et al.*, 2008; Hofheinz *et al.*, 2009) by applying control pulses to a qubit to put it into a superposition of the ground state $|g\rangle$ and the excited state $|e\rangle$

$$|\psi_{\text{initial}}\rangle = \alpha|g\rangle + \beta|e\rangle. \tag{2.60}$$

Allowing the qubit to spontaneously decay (if it is excited) leaves the qubit in the ground state and the electromagnetic field in a superposition of zero and one photon with coefficients α and β inherited from the qubit

$$|\psi_{\text{final}}\rangle = |g\rangle \left[\alpha|0\rangle + \beta|1\rangle\right]. \tag{2.61}$$

This operation maps a stationary qubit onto a 'flying qubit' (the photon) and is an essential step towards communicating quantum information via photons. In the experiment of Houck et al. (Houck *et al.*, 2007) the photons could be sent into a square law detector to measure the photon number, or into a homodyne detector to measure either quadrature of the electric field (equivalent to measuring $\hat{Q}$ or $\hat{\phi}$ in Eq. (2.42). The experiment directly showed that the one photon Fock state had zero electric field on average and that the phase of the electric field for superposition states was determined by the phase imposed initially upon the qubit superposition state. We tend to think of spontaneous emission as an incoherent process but the above results show that this is not entirely correct. What we really mean by incoherent is that the decay of an atom which starts purely in the excited state yields a photon state which varies randomly from shot to shot and which vanishes only on average.

In the UCSB experiments (Hofheinz *et al.*, 2008; Hofheinz *et al.*, 2009), complex superpositions of resonator Fock states were engineered and then measured via the their effect on the state of the qubit, rather than by homodyne measurement of the photon state.

2.3.1 Driven LC Oscillators

Before continuing, it is useful to return to the classical circuit analysis and think about how we should include a driving force on the oscillator. Returning to Fig. (2.2), let us consider adding a signal source to the circuit at the node labelled ϕ as shown in Fig. (2.4a). The first question we have to answer is whether we should use a voltage source or a current source. Ideally, the former has zero impedance and the latter has infinite impedance. A voltage source set to zero drive amplitude would short the ϕ node to ground and ruin the oscillator. Conversely, a current source set to zero drive amplitude would have no effect on the oscillator at all since the voltage oscillations would not be damped by the infinite impedance of the current source. Thus we should use a current source which will minimize the damping. [Generically resonators will be driven through a coupling capacitor or antenna structure connected to a $\sim 50\Omega$ transmission lines which will introduce some damping.] For the moment we will assume the drive is classical. (More on the meaning of classical further below.)

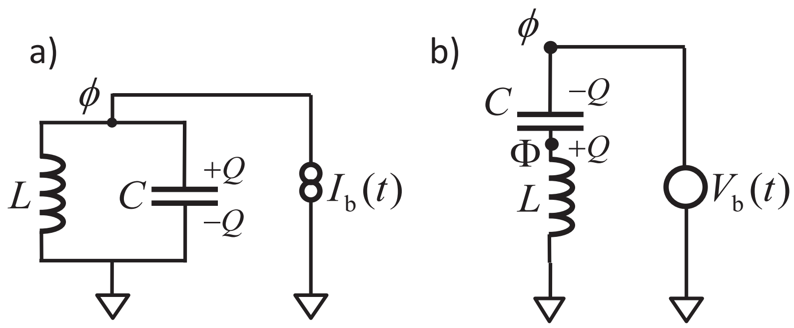

Fig. 2.4 (a) Parallel LC oscillator driven at the node ϕ by a classical external current source with infinite impedance. (b) Series LC oscillator driven at the node ϕ by a classical external voltage source with zero impedance.

Consider the following modification of the Lagrangian in Eq. (2.29)

$$\mathcal{L} = \frac{1}{2}C\dot{\phi}^2 - \frac{1}{2L}\phi^2 + I_b\phi, \tag{2.62}$$

where $I_b(t)$ is the (classical) time-dependent bias current delivered by the source. We can think of the third term as a Lagrange multiplier which enforces current conservation. From the Euler-Lagrange equation of motion

$$-\frac{d}{dt}\frac{\delta\mathcal{L}}{\delta\dot{\phi}} + \frac{\delta\mathcal{L}}{\delta\phi} = 0 \tag{2.63}$$

we obtain

$$\dot{Q} + \frac{\phi}{L} = I_{\mathrm{b}}(t), \tag{2.64}$$

which is simply the equation for current conservation at the ϕ node. Converting the Lagrangian to the classical Hamiltonian yields

$$H = \frac{Q^2}{2C} + \frac{\phi^2}{2L} - I_{\mathrm{b}}(t)\phi, \tag{2.65}$$

we see that the bias current acts as a force conjugate to the coordinate ϕ. We can view the current conservation equation above as Hamilton's equation of motion giving the time rate of change of the momentum in terms of the sum of the oscillator spring force plus the external force

$$\dot{Q} = -\frac{\phi}{L} + I_{\mathrm{b}}(t). \tag{2.66}$$

So far we have only considered the parallel LC resonator. We turn now to the series resonator illustrated in Fig. (2.4b). Clearly there can be no oscillations unless the node ϕ is connected to ground so that current can flow. This means that the series resonator should be driven by a zero impedance voltage source instead of a current source. The Lagrangian for this system is

$$\mathcal{L} = \frac{1}{2}C[\dot{\Phi} - \dot{\phi}]^2 - \frac{\Phi^2}{2L}, \tag{2.67}$$

from which it follows that the Hamiltonian is

$$H = \frac{Q^2}{2C} + \frac{\Phi^2}{2L} + V_{\mathrm{b}}(t)Q, \tag{2.68}$$

where $\dot{\phi} = V_{\mathrm{b}}(t)$ is fixed by the bias voltage. In this case, the external control parameter is the voltage rather than the drive current and the internal variable being controlled is the charge rather than the flux.

Exercise 2.3 Rederive the Lagrangian and the Hamiltonian for the series resonator shown in Fig. (2.4b) except with the capacitor and inductor interchanged so that the external voltage source is attached to the inductor rather than the capacitor. The physics should be identical to the previous case, but the mathematical expressions will look rather different. Can you find a change of coordinates that maps the problem back to the previous form?

2.3.2 Coherent States

Now that we understand the classically driven quantum harmonic oscillator, we are in a position to study coherent states of oscillation. A simple way to achieve a superposition of different number states in a quantum oscillator is to drive it with a classical external driving force so that the ground state is displaced and mapped to a so-called 'coherent state'

$$\Psi_0(\phi) \longrightarrow \Psi_\Delta(\phi) = \Psi_0(\phi - \Delta)). \tag{2.69}$$

Coherent states are discussed below and in further detail in Appendix E. In addition to having coherent states displaced in position, one can also have them displaced in momentum. These simply correspond to being in different parts of the classical oscillation cycle. We also discuss below what it means to have a 'classical' drive.

Using the Taylor series expansion to all orders we can write the unitary transformation that displaces the state as

$$\Psi_\Delta(\phi) = e^{-\Delta\frac{\partial}{\partial\phi}}\Psi_0(\phi) \tag{2.70}$$

$$= e^{-\frac{i}{\hbar}\Delta\hat{Q}}\Psi_0(\phi), \tag{2.71}$$

which illustrates the fact that the momentum $\hat{Q}$ is the generator of displacements of its conjugate coordinate ϕ. The unitary displacement operator may be written as

$$U_\alpha = e^{-\frac{i}{\hbar}\Delta\hat{Q}} = e^{-\alpha(\hat{a}-\hat{a}^\dagger)}, \tag{2.72}$$

where the dimensionless displacement parameter is

$$\alpha \equiv \frac{\Delta Q_{\mathrm{ZPF}}}{\hbar} = \frac{\Delta}{2\Phi_{\mathrm{ZPF}}}. \tag{2.73}$$

Now using the Feynman disentangling theorem (Mahan, 2000) derived in Appendix D, this can be normal ordered

$$U_\alpha = e^{+\alpha\hat{a}^\dagger}e^{-\alpha\hat{a}}e^{-\frac{1}{2}|\alpha|^2}. \tag{2.74}$$

Taking advantage of the fact that $\hat{a}|0\rangle = 0$, we see that in second-quantized notation the coherent state becomes

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger}|0\rangle \tag{2.75}$$

Exercise 2.4 Since U_α is unitary, it must be that $\langle\alpha|\alpha\rangle = 1$. Verify this by direct calculation from Eq. (2.75).

Coherent states have some very nice properties. For example, because they are special coherent superpositions of all possible photon numbers, they are eigenstates of the photon destruction operator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{2.76}$$

You can destroy a photon and still be in the same state! Curiously coherent states are *not* eigenstates of $\hat{a}^\dagger$. It is clear that $\hat{a}^\dagger|\alpha\rangle$ has no amplitude for zero photons and hence is linearly independent of $|\alpha\rangle$ (and therefore not an eigenstate). One can reach the same conclusion by noting that $\hat{a}$ and $\hat{a}^\dagger$ do not commute.

$$[\hat{a},\hat{a}^\dagger]|\alpha\rangle = |\alpha\rangle \neq 0. \tag{2.77}$$

On the other hand, it is true that the mean phonon number is given by

$$\bar{N} = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2. \tag{2.78}$$

The phonon number distribution in a coherent state is given by the standard Poisson distribution

$$P_n = |\langle n|\alpha\rangle|^2 = \frac{\bar{N}^n}{n!}e^{-\bar{N}}. \tag{2.79}$$

Exercise 2.5 Derive Eq. (2.76) and Eq. (2.79).

Because $\hat{a}^\dagger$ is a raising operator for the energy, the coherent state has a very simple time evolution even though it is itself not an energy eigenstate. The displacement parameter α becomes complex and its phase increases linearly in time. That is, the real and imaginary parts of α simply vary sinusoidally in time indicating that the displacement alternates between position and momentum:

$$|\alpha(t)\rangle = |e^{-i\Omega t}\alpha(0)\rangle = e^{-\frac{1}{2}|\alpha|^2}\, e^{\alpha e^{-i\Omega t}\hat{a}^\dagger}|0\rangle. \tag{2.80}$$

This corresponds in the classical limit to the circular motion in phase space of the simple harmonic oscillator.

Rather than working with $\hat{\phi}$ and $\hat{Q}$, we will find it convenient to work with the dimensionless quadrature amplitudes

$$\hat{X} \equiv \frac{1}{2}\left[\hat{a} + \hat{a}^\dagger\right] \tag{2.81}$$

$$\hat{Y} \equiv -i\frac{1}{2}\left[\hat{a} - \hat{a}^\dagger\right]. \tag{2.82}$$

These hermitian operators are effectively the real and imaginary parts of the $\hat{a}$. Like $\hat{\phi}$ and $\hat{Q}$, they are canonically conjugate with the following commutator

$$[\hat{X}, \hat{Y}] = +\frac{i}{2} \tag{2.83}$$

and for coherent states obey

$$\langle\alpha|\hat{X}|\alpha\rangle = \mathrm{Real}\,\alpha(t) \tag{2.84}$$

$$\langle\alpha|\hat{Y}|\alpha\rangle = \mathrm{Imag}\,\alpha(t) \tag{2.85}$$

$$\langle\alpha|[\hat{X} - \langle\hat{X}\rangle]^2|\alpha\rangle = \langle 0|[\Delta\hat{X}]^2|0\rangle = \frac{1}{4} \tag{2.86}$$

$$\langle\alpha|[\hat{Y} - \langle\hat{Y}\rangle]^2|\alpha\rangle = \langle 0|[\Delta\hat{Y}]^2|0\rangle = \frac{1}{4}. \tag{2.87}$$

The last two equations show that there are quantum fluctuations in $\hat{X}$ and $\hat{Y}$ (as there must be since they do not commute with each other). The resulting uncertainties in the measured values of these quantities play a central in understanding quantum noise (Clerk *et al.*, 2010). The energy of the oscillator (in units of $\hbar\Omega$) is

$$\hat{\epsilon} = \hat{X}^2 + \hat{Y}^2 = \hat{N} + \frac{1}{2}, \tag{2.88}$$

so the number operator is simply

$$\hat{N} = \hat{X}^2 + \hat{Y}^2 - \frac{1}{2}. \tag{2.89}$$

To understand the fluctuations in photon number, let us consider a coherent state with amplitude $\alpha = \sqrt{N}$ which is real. As illustrated in Fig. (2.5), fluctuations in $\hat{X}$

lead to photon number fluctuations (fluctuations in the length of the phasor) while fluctuations in $\hat{Y}$ lead to fluctuations in the phase of the coherent state as measured in homodyne detection (Clerk *et al.*, 2010). As we have seen, the coherent state is nothing more than a displaced vacuum state

$$|\alpha\rangle = U_\alpha|0\rangle. \tag{2.90}$$

Instead of actively displacing the physical system, we can equivalently leave the system alone and displace the coordinate system, transforming all operators according to the usual rule

$$\tilde{a} = U_\alpha^\dagger \hat{a} U_\alpha = \hat{a} + \alpha \tag{2.91}$$

$$\tilde{a}^\dagger = U_\alpha^\dagger \hat{a}^\dagger U_\alpha = \hat{a}^\dagger + \alpha^*. \tag{2.92}$$

Note that the analog of Eq. (2.76) is

$$\tilde{a}|0\rangle = \alpha|0\rangle. \tag{2.93}$$

We commonly refer to α as the classical amplitude of the motion and if $|\alpha| \gg 1$ it dominates over the quantum fluctuations around the classical value of the amplitude. As mentioned earlier, weakly coupling a system to an oscillator mode in a large amplitude coherent state produces what is effectively a classical drive with negligible quantum fluctuations. For example we might apply a force $\hat{F}$ to an oscillator whose coordinate is $\hat{y} = y_{\mathrm{ZPF}}(\hat{b} + \hat{b}^\dagger)$ via the coupling

$$\hat{V} = -\hat{F}\hat{y}. \tag{2.94}$$

For the case in which the force is supplied by linear coupling to a second 'drive' oscillator whose position operator is $\hat{x} = x_{\mathrm{ZPF}}(\hat{a} + \hat{a}^\dagger)$, the Hamiltonian would have the generic form

$$H = \omega_{\mathrm{R}}\hat{b}^\dagger\hat{b} + \omega_{\mathrm{d}}\hat{a}^\dagger\hat{a} + g(\hat{a} + \hat{a}^\dagger)(\hat{b} + \hat{b}^\dagger). \tag{2.95}$$

Changing to a frame rotating with the drive oscillator via the unitary transformation

$$\hat{U} = e^{+i\omega_{\mathrm{d}}t\hat{a}^\dagger\hat{a}} \tag{2.96}$$

the Hamiltonian becomes

$$H_1 = UHU^\dagger + U[-i\frac{d}{dt}, U^\dagger] = \omega_{\mathrm{R}}\hat{b}^\dagger\hat{b} + g\left(e^{-i\omega_{\mathrm{d}}t}\hat{a} + e^{-i\omega_{\mathrm{d}}t}\hat{a}^\dagger\right)(\hat{b} + \hat{b}^\dagger). \tag{2.97}$$

If the drive oscillator is initially placed in a high amplitude coherent state it is convenient to make the displacement transformation in Eq. (2.92) to obtain the transformed coupling Hamiltonian

$$H_2 = \omega_{\mathrm{R}}\hat{b}^\dagger\hat{b} + g\left(e^{-i\omega_{\mathrm{d}}t}\alpha + e^{-i\omega_{\mathrm{d}}t}\alpha^*\right)(\hat{b} + \hat{b}^\dagger) + H_{\mathrm{Q}}. \tag{2.98}$$

We see in the first two terms that the system oscillator is quantum and subject to a classical drive. The last term describes the quantum fluctuations associated with the drive

$$H_{\mathrm{Q}} = g\left(e^{-i\omega_{\mathrm{d}}t}\hat{a} + e^{-i\omega_{\mathrm{d}}t}\hat{a}^\dagger\right)(\hat{b} + \hat{b}^\dagger). \tag{2.99}$$

Because (initially at least) the drive oscillator is now in the ground state (in the new frame), the quantum fluctuations of the drive are small compared to the classical part, if (in the original frame) the drive amplitude corresponds to a state with many quanta: $\bar{n} = |\alpha|^2 \gg 1$. This will continue to remain true over time provided that that the drive strength $g|\alpha|$ and the detuning $\omega_{\mathrm{d}} - \omega_{\mathrm{R}}$ are such that the number of quanta transferred from the drive to the system via the action of H_{Q} remains much smaller than $\bar{n}$.

A good example of this physics is provided by a two-port resonator with one weakly coupled port and one strongly coupled port. The damping of the resonator will be controlled by the port strongly coupled to the environment since most photons will escape through that port. If the system is continuously driven at the weakly coupled port, most photons from the drive line will be reflected, so a relatively large coherent drive from a microwave signal generator is required to excite the resonator cavity. This corresponds to the limit described above of small g and large α for which the classical approximation is valid. All we require is that the power in the incoming drive wave be mostly reflected so that it greatly exceeds the power emitted by the driven resonator from its strongly coupled port. In the theory of parametric amplifiers, this is known as the 'stiff pump' limit. No matter what the driven system does, the pump amplitude stays fixed and essentially classical.

Exercise 2.6 Derive Eqs. (2.91-2.92) by differentiating with respect to α and solving the resulting differential equation.

Exercise 2.7 Solve the Heisenberg equation of motion for $\hat{b}$ using the Hamiltoninan in Eq. (2.98) but neglecting the quantum fluctuation term H_{Q}. Show that this classical drive applied to an oscillator initially in a coherent state (including possibly the vacuum state) always leaves the system in a coherent state.

Exercise 2.8 Show by direct computation that for the Bose-Einstein number distribution for a thermal photon state

$$\langle\langle [\hat{N} - \bar{N}]^2 \rangle\rangle = \bar{N}(\bar{N} + 1). \tag{2.100}$$

If you are familiar with Wick's theorem, use that to achieve the same result.

Exercise 2.9 $\hat{X}^2$ and $\hat{Y}^2$ are clearly Hermitian operators with non-negative eigenvalues. How then can you explain the fact that

$$\langle 0|\hat{X}^2\hat{Y}^2|0\rangle = -\frac{1}{16} \tag{2.101}$$

is negative? Similarly how can

$$\langle 0|\hat{X}\hat{Y}|0\rangle = \frac{i}{4} \tag{2.102}$$

be complex?

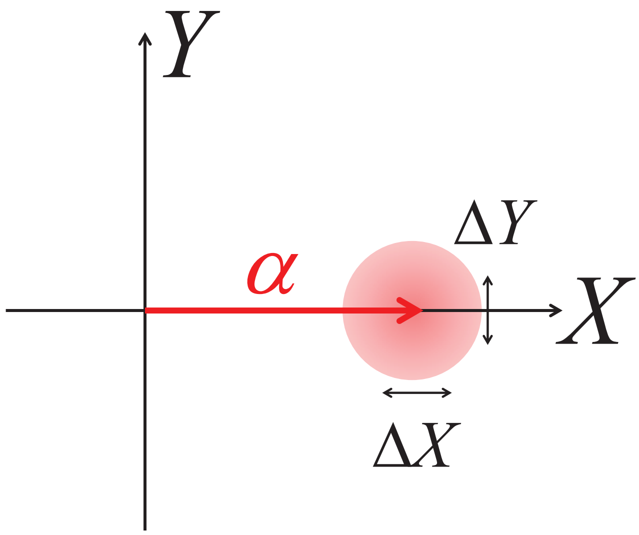

Fig. 2.5 Quantum fluctuations of amplitude and phase quadratures in a coherent state $|\alpha\rangle$.

Writing the X quadrature amplitude as

$$\hat{X} = \alpha + \Delta\hat{X}, \tag{2.103}$$

we see that $\Delta\hat{X}$ has the same statistical properties in the coherent state $|\alpha\rangle$ as $\hat{X}$ does in the vacuum state. The number fluctuations are therefore given by the usual Poisson distribution result derived above

$$\langle\alpha|[\hat{N} - \bar{N}]^2|\alpha\rangle = \langle\alpha|\left[2\alpha\Delta\hat{X} + \Delta\hat{X}^2 + \Delta\hat{Y}^2 - \frac{1}{2}\right]^2|\alpha\rangle = \bar{N}. \tag{2.104}$$

Essentially the above results mean that a coherent laser or microwave beam is as classical as possible. The fluctuations come only from the fact that the photon detection events are discrete and the photons are sprinkled randomly throughout the beam in an uncorrelated manner. A thermal beam has larger fluctuations because the photons tend to bunch together (Clerk *et al.*, 2010).

Fluctuations in the quadrature orthogonal to α cause uncertainty in a measurement of the phase of the coherent state. For the case of α real and in the limit $|\alpha| \gg 1$, we have[6]

$$\Delta\hat{\theta} \approx \frac{\Delta\hat{Y}}{\alpha} \tag{2.105}$$

and

$$\langle\alpha|(\Delta\hat{\theta})^2|\alpha\rangle = \frac{1}{4\bar{N}}. \tag{2.106}$$

[6] The ‘phase’ operator defined here does not have the angular periodicity of a phase and is only valid for small angles.

Thus we arrive at the fundamental number-phase uncertainty relation

$$\langle\alpha|(\Delta\hat{\theta})^2|\alpha\rangle^{1/2}\langle\alpha|(\Delta\hat{N})^2|\alpha\rangle^{1/2} \geq \frac{1}{2}. \tag{2.107}$$

Coherent states are minimum uncertainty gaussian states which satisfy this relation as an equality. Other non-gaussian states satisfy this relation only as an inequality.

From the equation of motion of the free oscillator we see that the quadrature amplitudes obey

$$\hat{X}(t) = \cos(\Omega t)\hat{X}(0) + \sin(\Omega t)\hat{Y}(0) \tag{2.108}$$
$$\hat{Y}(t) = \cos(\Omega t)\hat{Y}(0) - \sin(\Omega t)\hat{X}(0) \tag{2.109}$$

In Appendix B we study photons traveling in transmission lines and we again find that the traveling modes are also harmonic oscillators. The above results provide the first hint that the $\sin\Omega t$ and $\cos\Omega t$ quadratures of a quantum electrical signal are canonically conjugate and hence cannot be simultaneously measured with perfect accuracy. Equivalently even the vacuum contains noise which will appear in any measurement in which one attempts to measure both quadratures of the signal. Eq. (2.87) tells us that this uncertainty gives a vacuum 'noise energy' (noise power per unit measurement bandwidth) of half a photon (Clerk *et al.*, 2010).

Exercise 2.10 Think through the above statement about noise energy at the classical level. Consider a noise source which is white (i.e., with constant spectral density S) over some large interval. Passing this noise through a filter which transmits a small bandwidth B centered on frequency ω will yield a power of $P = SB$. The wider the bandpass the more power. Thus we see that the spectral density is power per unit bandwidth which has units of energy. For a quantum thermal source feeding a photomultiplier (which measures $\hat{a}^\dagger\hat{a}$), this is $S = \hbar\omega\bar{N}$ and we say that 'the noise energy is $\bar{N}$ photons.' A photomultiplier feed by vacuum noise has zero output. However listening to the vacuum noise power through a phase preserving amplifier or (equivalently) using a heterodyne detector which measures the power in the quadrature amplitudes $\langle\hat{X}^2 + \hat{Y}^2\rangle = \frac{1}{2}$ yields a noise energy of half a photon (Clerk *et al.*, 2010).

2.4 Coupled LC Resonators

Having thoroughly analyzed the simple LC oscillator, it is a useful exercise to consider how to quantize a pair of LC oscillators connected by a coupling capacitor as shown in Fig. (2.6). This will teach us how to handle slightly more complex circuits and will set the stage for understanding the coupling of a qubit to a microwave resonator.

Choosing the fluxes Φ_1 and Φ_2 as the coordinates of the two oscillators, the Lagrangian can be written

$$\mathcal{L} = \frac{1}{2}C_1\dot{\Phi}_1^2 + \frac{1}{2}C_2\dot{\Phi}_2^2 + \frac{1}{2}C_0[\dot{\Phi}_1 - \dot{\Phi}_2]^2 - \frac{1}{2L_1}\Phi_1^2 - \frac{1}{2L_2}\Phi_2^2 \tag{2.110}$$

It is convenient to use a matrix notation

$$\mathcal{L}\frac{1}{2}\dot{\Phi}C\dot{\Phi} - \frac{1}{2}\Phi L^{-1}\Phi, \tag{2.111}$$

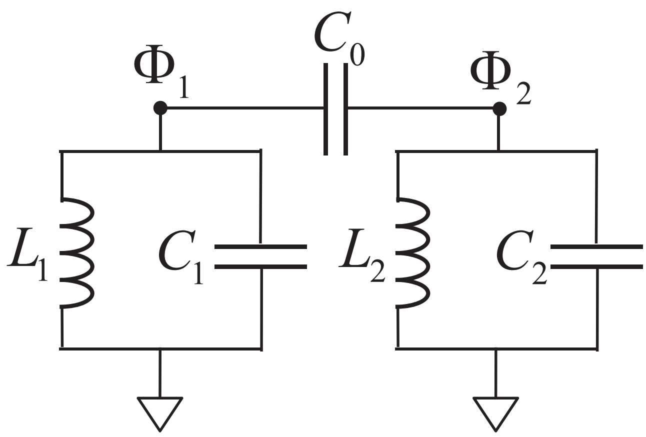

Fig. 2.6 A pair of LC oscillators connected by coupling capacitor C_0.

where the capacitance matrix is

$$C \equiv \begin{pmatrix} C_1 + C_0 & -C_0 \\ -C_0 & C_2 + C_0 \end{pmatrix}, \tag{2.112}$$

and the inverse inductance matrix is

$$L^{-1} \equiv \begin{pmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{pmatrix}. \tag{2.113}$$

At this point there are two ways to proceed, which are described below.

METHOD I: FIND THE HAMILTONIAN, THEN DIAGONALIZE. In the first method we will use the given coordinates to find the canonical momenta and from there construct the Hamiltonian which will contain a coupling between the two oscillators.

The canonical momenta are given by

$$Q_i \equiv \frac{\delta \mathcal{L}}{\delta \dot{\Phi}_i} = C_{ij}\dot{\Phi}_j, \tag{2.114}$$

where we employ the Einstein summation convention for repeated indices. In terms of the inverse of the capacitance matrix we have

$$\dot{\Phi} = C^{-1}Q. \tag{2.115}$$

The Hamiltonian $H = Q_i\dot{\Phi}_i - \mathcal{L}$ now takes the canonical form

$$H = \frac{1}{2}QC^{-1}Q + \frac{1}{2}\Phi L^{-1}\Phi. \tag{2.116}$$

The inverse of the capacitance matrix is

$$C^{-1} = \frac{1}{C_1C_2 + C_0C_1 + C_0C_2}\begin{pmatrix} C_2 + C_0 & +C_0 \\ +C_0 & C_1 + C_0 \end{pmatrix}, \tag{2.117}$$

It is useful to define two frequencies and a coupling constant:

$$\omega_j^2 =\equiv \frac{1}{L_j}\left(C^{-1}\right)_{jj}, \tag{2.118}$$

and

$$\beta \equiv \frac{C_0}{\sqrt{(C_1 + c_0)(C_2 + C_0)}}, \tag{2.119}$$

which yields

$$C^{-1} = \begin{pmatrix} L_1\omega_1^2 & +\beta\sqrt{L_1L_2}\omega_1\omega_2 \\ \beta\sqrt{L_1L_2}\omega_1\omega_2 & L_2\omega_2^2 \end{pmatrix}. \tag{2.120}$$

We can now write the Hamiltonian $H = H_0 + V$ in terms of two oscillators with masses L_j and coupled through their momenta

$$H_0 = \frac{1}{2}L_1\omega_1^2Q_1^2 + \frac{1}{2L_1}\Phi_1^2 + \frac{1}{2}L_2\omega_1^2Q_2^2 + \frac{1}{2L_2}\Phi_2^2 \tag{2.121}$$

$$V = \beta\sqrt{L_1L_2}\omega_1\omega_2Q_1Q_2. \tag{2.122}$$

We quantize as usual by converting to operators with the canonical commutation relation

$$[\hat{Q}_i, \hat{\Phi}_j] = -i\hbar\delta_{ij}. \tag{2.123}$$

Defining creation and annihilation operators in the usual way we have

$$H_0 = \sum_{j=1}^{2}\hbar\omega_j\left(\hat{a}_j^\dagger\hat{a}_j + \frac{1}{2}\right) \tag{2.124}$$

$$V = -\beta\hbar\sqrt{\omega_1\omega_2}(\hat{a}_1 - \hat{a}_1^\dagger)(\hat{a}_2 - \hat{a}_2^\dagger), \tag{2.125}$$

which can be diagonalized via a Bogoljubov transformation.

Exercise 2.11 Find the Bogoljubov transformation which diagonalizes $H_0 + V$ defined above.

METHOD II: DIAGONALIZE THE LAGRANGIAN, THEN THE HAMILTONIAN. The first method used the original coordinates and found their canonical momenta and from there constructed the (non-diagonal) Hamiltonian. In the second method, we will find the normal mode coordinates which diagonalize the Lagrangian. In terms of these, the Hamiltonian will be automatically diagonal.

When we try to diagonalize the Lagrangian in Eq. (2.111), we are faced with the problem that the capacitance and inductance matrices do not commute and hence cannot be simultaneously diagonalized by a unitary transformation. We can cure this problem by making a *similarity transformation* which maps L^{-1} to the identity matrix. We simply choose scaled coordinates

$$\psi_j = \frac{1}{\sqrt{L_j}}\Phi_j. \tag{2.126}$$

In terms of these the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\dot{\psi}_i A_{ij}\dot{\psi}_j - \frac{1}{2}\psi_i \delta_{ij}\psi_j \tag{2.127}$$

where

$$A \equiv \begin{pmatrix} \frac{1}{\Omega_1^2} & -\frac{\beta}{\Omega_1\Omega_2} \\ -\frac{\beta}{\Omega_1\Omega_2} & \frac{1}{\Omega_2^2} \end{pmatrix}, \tag{2.128}$$

where we define frequencies (different from the previous method)

$$\frac{1}{\Omega_1^2} \equiv L_1(C_1 + C_0) \tag{2.129}$$

$$\frac{1}{\Omega_2^2} \equiv L_2(C_2 + C_0). \tag{2.130}$$

Since A commutes with the identity matrix, we can now proceed as usual to perform a rotation among the coordinates to diagonalize the Lagrangian. Let S be the orthogonal transformation that diagonalizes A. The normal modes and eigenvalues are then given by

$$\tilde{\psi} = S\psi \tag{2.131}$$

$$\tilde{A} = \begin{pmatrix} \frac{1}{\tilde{\Omega}_1^2} & 0 \\ 0 & \frac{1}{\tilde{\Omega}_2^2} \end{pmatrix} = SAS^{\mathrm{T}}. \tag{2.132}$$

Exercise 2.12 Find the normal modes and eigenfrequencies above. Hint: Write $A = \bar{A} + Z\sigma^z + X\sigma^x$ and think of it as a spin problem which has eigenvalues

$$\epsilon_\pm = \bar{A} \pm \sqrt{X^2 + Z^2} \tag{2.133}$$

and eigenfunctions which follow from

$$S = \begin{pmatrix} +\cos\frac{\theta}{2} & +\sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & +\cos\frac{\theta}{2} \end{pmatrix}, \tag{2.134}$$

where $\tan\theta = X/Z$.

2.5 Modes of Transmission Lines Resonators

The above lengthy discussion of the simple harmonic oscillator has laid the very important groundwork for our next topic which is the quantum modes of transmission lines. We will start with finite length transmission lines which have discrete electromagnetic resonances, each of which will turn out to be an independent simple harmonic oscillator. Then we will move on to the semi-infinite transmission line and discover that it can act like a dissipative bath even though every one of its electrical elements is non-dissipative.

Our finite length transmission line could be a length of ordinary coaxial cable or its 2D equivalent, the coplanar waveguide (CPW), which consists of a superconducting wire evaporated on an insulating substrate and having superconducting ground planes adjacent to it on the same surface as shown in Fig. (2.7). Such a system exhibits many standing wave resonances and we will soon see that each resonance is an independent harmonic oscillator equivalent to the simple LC oscillator just discussed. The discretized equivalent circuit for the CPW resonator is also shown in Fig. (2.7). In our initial analysis we will neglect the presence of the qubit and neglect the capacitors C_0 at each end which couple the resonator to the external transmission lines. We can thus assume in this first example open-circuit boundary conditions for which the current (but not the voltage) vanishes at the ends of the resonator.

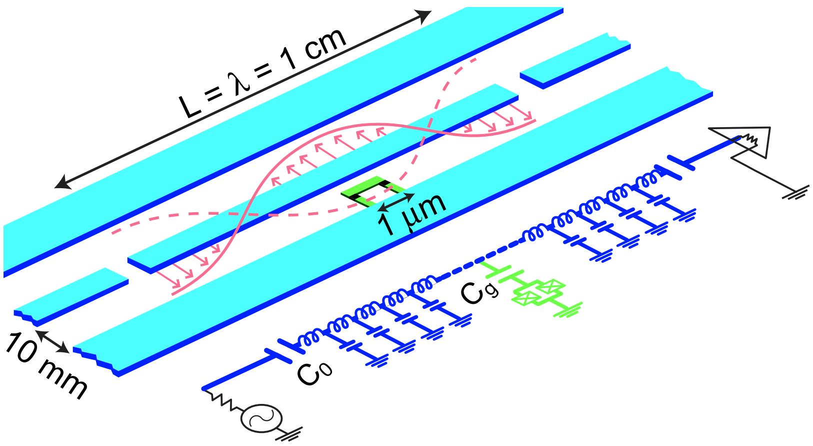

Fig. 2.7 Schematic illustration of a typical coplanar waveguide (CPW) resonator used in circuit QED together with its discretized lumped-element equivalent circuit. The qubit lies between the center pin and the adjacent ground plane and is located at an antinode of the electric field, shown in this case for the full-wave resonance of the CPW. From (Blais *et al.*, 2004).

It is convenient to define a flux variable analogous to that used above but now dependent on position (Devoret, 1997)

$$\Phi(x,t) \equiv \int_{-\infty}^{t} d\tau\, V(x,\tau), \tag{2.135}$$

where $V(x,t) = \partial_t \Phi(x,t)$ is the local voltage on the transmission line at position x and time t. The inductance and capacitance per unit length are ℓ and c respectively. Each segment of the line of length dx has inductance $\ell\, dx$ and the voltage drop along it is $-dx\, \partial_x \partial_t \Phi(x,t)$. The flux through this inductance is thus $-dx\, \partial_x \Phi(x,t)$ and the local value of the current is given by the constitutive equation

$$I(x,t) = -\frac{1}{\ell}\, \partial_x \Phi(x,t). \tag{2.136}$$

The Lagrangian for a system of length L (L is not to be confused with some discrete inductance)

$$\mathcal{L}_g \equiv \int_0^L dx\, \mathcal{L}(x,t) = \int_0^L dx\, \left[\frac{c}{2}(\partial_t \Phi)^2 - \frac{1}{2\ell}\,(\partial_x \Phi)^2\right], \tag{2.137}$$

The Euler-Lagrange equation for this Lagrangian is simply the wave equation

$$v_{\mathrm{p}}^2 \partial_x^2 \Phi - \partial_t^2 \Phi = 0. \tag{2.138}$$

The momentum conjugate to $\Phi(x)$ is simply the charge density

$$q(x,t) \equiv \frac{\delta \mathcal{L}_g}{\delta \partial_t \Phi} = c\partial_t \Phi = cV(x,t) \tag{2.139}$$

and so the Hamiltonian is given by

$$H = \int_0^L dx\, \left\{\frac{1}{2c} q^2 + \frac{1}{2\ell}(\partial_x \Phi)^2\right\}. \tag{2.140}$$

Let us next proceed to consider the classical normal mode solutions of Eq. (2.138). If we assume a sinusoidal time-dependence with angular frequency ω,

$$\Phi(x,t) = e^{-i\omega t}\phi(x), \tag{2.141}$$

we arrive at the Schrödinger like eigenvalue problem

$$-\partial_x^2 \phi(x) = k^2 \phi(x), \tag{2.142}$$

where $k = \omega/v_{\mathrm{p}}$ and the mode wave velocity is $v_{\mathrm{p}} = \frac{1}{\sqrt{\ell c}}$. The open-circuit (zero-current) boundary conditions tell us that the eigenfunctions have vanishing derivative at the boundaries. We choose a particular normalization for eigenfunctions which will keep the equations looking as close to those of the single harmonic oscillator as possible

$$\phi_n(x) = \sqrt{2}\cos(k_n x), \tag{2.143}$$

where $n \,\epsilon\, \{0, 1, 2, 3, \ldots\}$, $k_n = \frac{n\pi}{L}$. Because for these boundary conditions the operator ∂_x^2 is self-adjoint, and because the eigenvalues are non-degenerate, the eigenfunctions have two helpful properties

$$\int_0^L dx\, \phi_n(x)\phi_m(x) = L\delta_{nm} \tag{2.144}$$

$$\int_0^L dx\, [\partial_x\phi_n(x)][\partial_x\phi_m(x)] = Lk_n^2\delta_{nm}. \tag{2.145}$$

From this it follows that the Lagrangian can be diagonalized using these (spatial) normal modes as a basis. Let us parameterize the field $\Phi(x,t)$ by

$$\Phi(x,t) = \sum_{n=0}^{\infty} \xi_n(t)\phi_n(x), \tag{2.146}$$

where the ξ_n are *arbitrary* (i.e. not necessarily sinusoidal) functions of time. Substituting into the Eq. (2.137) and using Eqs. (2.144-2.145)

$$\mathcal{L}_g = \frac{1}{2}Lc\sum_{n=0}^{\infty}[\partial_t\xi_n]^2 - \omega_n^2\xi_n^2 \tag{2.147}$$

we see that each normal mode becomes an independent simple harmonic oscillator. The momentum conjugate to the normal mode amplitude ξ_n is

$$q_n = \frac{\delta\mathcal{L}_g}{\delta\partial_t\xi_n} = Lc\partial_t\xi_n, \tag{2.148}$$

so the Hamiltonian is

$$H = \frac{1}{2}\sum_{n=0}^{\infty}\left\{\frac{1}{Lc}q_n^2 + Lc\omega_n^2\xi_n^2\right\}, \tag{2.149}$$

which we can quantize as before. Before doing so, let us note that the $n=0$ mode is a 'free particle' rather than a harmonic oscillator because its spring constant vanishes. This mode simply corresponds to a uniform net charge distributed evenly along the transmission line. For a free particle the momentum (in this case charge) is a constant and the coordinate (flux) increases linearly with time. In most situations the total charge is indeed simply a constant of the motion (and typically vanishes) and we can ignore the zero mode altogether. We will assume this is the case henceforth.

We end up with a set of independent normal modes with coordinate ξ_n and conjugate momentum q_n which when quantized can be expressed in terms of mode raising and lowering operators in a manner analogous to Eq. (2.42)

$$\hat{\xi}_n = \sqrt{\frac{\hbar}{2\omega_n Lc}}(\hat{a}_n + \hat{a}_n^\dagger) \tag{2.150}$$

$$\hat{q}_n = -i\sqrt{\frac{\hbar\omega_n Lc}{2}}(\hat{a}_n - \hat{a}_n^\dagger) \tag{2.151}$$

where the ladder operators of the different modes obey

$$[\hat{a}_n, \hat{a}_m^\dagger] = \delta_{nm}. \tag{2.152}$$

Note that, just as in the single mode case in Eq. (2.42), there is a certain arbitrariness in the choice of the phase of the destruction operators (which can be independently varied for each separate mode).

If we are coupling a qubit to a resonator at some particular position x, we need to be able to express the flux and charge density operators at that point in terms of the normal mode operators. Eq. (2.146) is readily extended to the quantum operators

$$\hat{\Phi}(x) = \sum_{n}^{\infty} \phi_n(x)\hat{\xi}_n, \tag{2.153}$$

as is Eq. (2.139)

$$\hat{q}(x) = \frac{1}{L}\sum_{n}^{\infty} \phi_n(x)\hat{q}_n. \tag{2.154}$$

Similarly, the analog of Eq. (2.53) for the voltage operator at point x is given by

$$\hat{V}(x) = \frac{1}{c}\hat{q}(x) = \frac{1}{L}\sum_{n=0}^{\infty} \phi_n(x)\hat{q}_n = -i\sum_{n=0}^{\infty}\sqrt{\frac{\hbar\omega_n}{2Lc}}(\hat{a}_n - \hat{a}_n^\dagger)\phi_n(x). \tag{2.155}$$

The total capacitance to ground of the resonator, Lc, enters this expression in a way that is similar to lumped element oscillator expression in Eq. (2.53). (Recall that L is the length of the resonator, not the inductance.)

Notice that the flux and charge density operators obey the following commutation relation

$$[\hat{q}(x), \hat{\Phi}(x')] = -i\hbar\frac{1}{L}\sum_{n}^{\infty} \phi_n(x)\phi_n(x'). \tag{2.156}$$

Using the completeness relation (and recalling that the factor of L appears because we did not normalize the eigenfunctions to unity) we end up with the standard field theoretic relation

$$[\hat{q}(x'), \hat{\Phi}(x)] = -i\hbar\delta(x - x'). \tag{2.157}$$

Expressing the quantum Hamiltonian in Eq. (2.140) in terms of these operators, we have simply

$$\hat{H} = \int_0^L dx\, \left\{\frac{1}{2c}\hat{q}^2 + \frac{1}{2\ell}(\partial_x\hat{\Phi})^2\right\}. \tag{2.158}$$

As a 'sanity check' let us look at the Hamilton equations of motion. Using commutation relation in Eq. (2.157) and its extension to

$$[\hat{q}(x'), \partial_x\hat{\Phi}(x)] = -i\hbar\partial_x\delta(x - x'). \tag{2.159}$$

we arrive at

$$\partial_t\hat{\Phi}(y) = \frac{i}{\hbar}[\hat{H}, \hat{\Phi}(y)] = \frac{1}{c}\hat{q}(y) \tag{2.160}$$

$$\partial_t\hat{q}(y) = \frac{i}{\hbar}[\hat{H}, \hat{q}(y)] = \frac{1}{\ell}\partial_y^2\hat{\Phi}(y). \tag{2.161}$$

and hence the quantum version of the wave equation in Eq. (2.138)

$$v_{\mathrm{p}}^2\partial_x^2\hat{\Phi}(x) - \partial_t^2\hat{\Phi}(x) = 0. \tag{2.162}$$

When we studied coherent states of a single oscillator we found that they were simply the vacuum state displaced in either position (flux) and/or momentum (charge).

For a multi-mode resonator one can coherently displace a linear combination of the of the normal modes. The familiar problem of solving the time evolution of a plucked string is a good classical analog. Suppose that we wish to displace the resonator degrees of freedom so that the local displacement obeys

$$\langle \hat{\Phi}(x) \rangle = \Delta(x), \tag{2.163}$$

where Δ is some specified function. The analog of Eq. (2.72) is simply

$$U_\Delta = e^{-\frac{i}{\hbar}\int_0^L dx\, \Delta(x)\hat{q}(x)}. \tag{2.164}$$

a form which is familiar from the theory of the Luttinger liquid (Kane and Fisher, 1992*b*; Kane and Fisher, 1992*a*). Using Eq. (2.154) this can be understood in terms of coherent displacement of each of the normal modes

$$U_\Delta = e^{-\frac{i}{\hbar}\sum_n \Delta_n \hat{q}_n} = \prod_n e^{-\frac{i}{\hbar}\Delta_n \hat{q}_n}. \tag{2.165}$$

Exercise 2.13 In analogy with Eqs. (2.91-2.92) show that

$$U_\Delta^\dagger \hat{\Phi}(y) U_\Delta = \hat{\Phi}(y) + \Delta(y). \tag{2.166}$$

Hint: It may be useful to scale $\Delta(x)$ by an overall factor θ and differentiate with respect to θ.

2.6 'Black Box' Quantization of Linear Circuits

We have so far studied a single LC oscillator and found that its quantum excitation energy $\hbar\Omega$ is given directly by its classical frequency Ω. We also found in Eq. (2.50) that the characteristic impedance $Z = \sqrt{L/C}$ determines the size of the zero-point fluctuations in flux and charge. The typical circuit that we will study is more complex than a single LC oscillator and might even be a 'black box' whose properties we need to determine. Suppose that we have such a black box and we have access to one port of this structure as shown in Fig. (2.8a). The only thing we know (or assume) is that all the elements inside the black box are linear and purely reactive; i.e., the black box is a network of inductors and capacitors. It might for example be a transmission line resonator such as we studied above. We may ultimately want to connect a qubit or some measurement apparatus to the port of the black box. In order to predict the quantum properties we need to know each of the normal modes of the box and the size of their zero-point fluctuations as seen at the port. Some modes may be localized inside the box and have very little amplitude at the port. Others may be more strongly coupled to the port.

Since the black box is linear, we can probe it by applying a sinusoidal drive and measuring the response. The are two ways to do this. First, one can hook up a current source which forces current

$$I(t) = i[\omega]e^{j\omega t} + i^*[\omega]e^{-j\omega t} \tag{2.167}$$

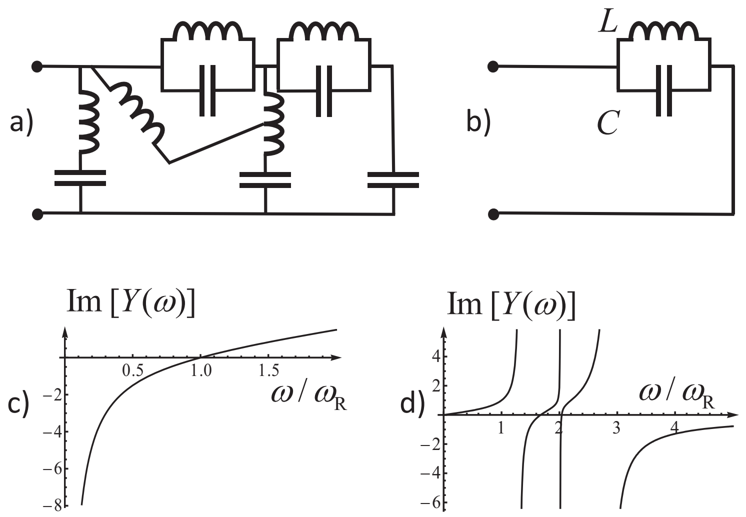

Fig. 2.8 a) One-port black box containing an arbitrary reactive network. b) Lumped element LC resonator. c) Imaginary part of the admittance of the LC resonator in (b) vs. dimensionless frequency showing that the admittance passes through zero with positive slope at the resonance frequency. d) Imaginary part of the admittance of a multi-resonance circuit with a capacitor in the input line similar to (a). Notice that the slope of the admittance at each of the zeros is different, corresponding to different characteristic impedances of the resonances.

through the circuit.[7] The linear response of the circuit is determined by measuring the resulting voltage at the input port

$$V(t) = v[\omega]e^{j\omega t} + v^*[\omega]e^{-j\omega t} \tag{2.168}$$

The linear response coefficient that relates the voltage response to the drive current is known as the impedance

$$v[\omega] = Z[\omega]i[\omega]. \tag{2.169}$$

Because the box contains only reactive elements (assumed finite in number) the impedance is purely imaginary. The poles of $Z[\omega]$ determine the eigenfrequencies of the circuit for which natural oscillations can occur without external input (when the input port is open circuited). Note that this is consistent with the fact that an ideal current

[7]To avoid confusion with the current i we follow the electrical engineering convention of using $j = -\sqrt{-1}$. In addition to avoid confusion between some function of time and its Fourier transform, we will use the convention that Fourier transformed quantities have the frequency argument in square brackets.

source has infinite internal impedance and hence drives the circuit while effectively keeping the input port open-circuited. The circuit presented in Fig. (2.9a) is a natural representation of an arbitrary frequency-dependent impedance[8]. It is important to understand that in general, the circuit elements used in this mathematical representation have no direct correspondence with any of the physical elements in the actual circuit. Note that if there is a pole in the impedance at zero frequency, it corresponds to the 'free-particle' Hamiltonian of a capacitor, $H = \frac{\hat{Q}^2}{2C}$ in series with the input (not shown in Fig. (2.9a).

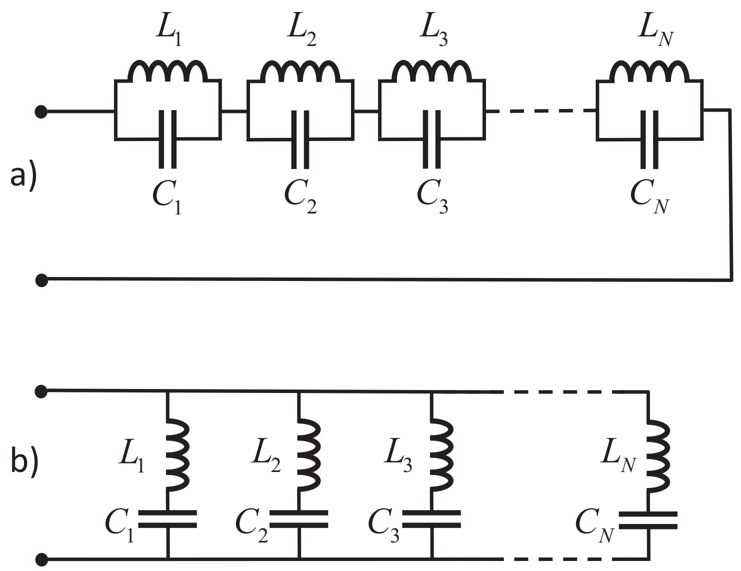

Fig. 2.9 a) Natural representation of an arbitrary impedance (assuming for simplicity that the impedance vanishes at zero frequency). The jth pole of the impedance occurs at the frequency of the jth collective mode $\omega_j = 1/\sqrt{L_j C_j}$ and can be detected by using an infinite-impedance current source to inject RF current into the input port and measuring the resulting RF voltage across across the port. b) Natural representation of an arbitrary admittance (assuming for simplicity that the admittance vanishes at zero frequency). The poles of the admittance determine the natural oscillation frequencies of the circuit when its input is shorted. These can be detected by using a zero-impedance RF voltage source to put a drive voltage across the input port and measuring the resulting RF current that flows into the port.

[8]Note that this particular representation has the property that there is a dc connection through all the inductors to ground. Hence the impedance vanishes at zero frequency. If this is not the case for the physical circuit, then we must include a series capacitor in the input line. This would be necessary for example to represent the impedance of the circuit shown in Fig. (2.8a).

The second way to measure the linear response is to attach a zero-impedance voltage source to the input and measure the resulting current response. The linear response coefficient that relates the current response to the voltage drive is known as the admittance

$$i[\omega] = Y[\omega]v[\omega] \tag{2.170}$$

which is simply the inverse of the impedance

$$Y[\omega] = Z^{-1}[\omega]. \tag{2.171}$$

The circuit presented in Fig. (2.9b) is a natural representation of an arbitrary frequency-dependent admittance. The poles of the admittance determine the natural oscillation frequencies of the circuit when its input port is short-circuited. Again, this is consistent with the excitation of these modes, this time using a zero-impedance voltage source. To reiterate, the poles of the admittance (zeros of the impedance) correspond to effective *s*eries LC resonances which would occur if the input port were short-circuited. These can be important but for the particular case where nothing is hooked up to the external port, these poles do not correspond to active degrees of freedom. An inductor and capacitor in series cannot oscillate on their own at non-zero frequencies unless the circuit is closed at the input port. Finally, we note that according to Foster's theorem (Foster, 1924), the (imaginary) admittance of a reactive circuit always passes through zero with positive slope so therefore each zero must be separated from the next by a pole as shown in Fig. (2.8d).

Physically, poles of response functions are the most natural thing to consider. However in numerical simulations, zeros are sometimes mathematically easier for a computer to handle than poles. Hence it can be convenient to work with the impedance representation in Fig. (2.9a) but numerically ascertain the zero-crossings of the admittance rather than the poles of the impedance.

As an example, suppose that the black box contains a single parallel LC oscillator as shown in Fig. (2.8b). Then the admittance is simply

$$Y[\omega] = j\omega C + \frac{1}{j\omega L} = \frac{+j}{Z_0}\left(\frac{\omega}{\omega_{\mathrm{R}}} - \frac{\omega_{\mathrm{R}}}{\omega}\right), \tag{2.172}$$

where $Z_0 \equiv \sqrt{\frac{L}{C}}$ is the characteristic impedance of the resonance. Note that this is indeed purely imaginary and further that it passes through zero at the resonance frequency $\Omega = \frac{1}{\sqrt{LC}}$ as shown in Fig. (2.8c). The admittance is zero because the inductor and capacitor have opposite admittances at the resonance frequency. But this is precisely the condition for self-sustaining oscillation where the currents in the inductor and capacitor are opposite to each other and no external input is needed.

It turns out that knowing the admittance (or impedance) of the box port as a function of frequency completely characterizes the classical and the quantum properties of the black box, as long as it contains only linear elements(Manucharyan *et al.*, 2007). We have already seen a hint of this in Eqs. (2.43-2.44) where we learned that the characteristic impedance of a resonance determines the zero-point fluctuations of the charge and flux degrees of freedom. Of course, knowing the frequency of an oscillator we can immediately write down the quantum Hamiltonian (neglecting the zero-point energy)

$$H_0 = \hbar\Omega\hat{a}^\dagger\hat{a}. \tag{2.173}$$

This is not enough however. If we couple an external circuit to our black box we need to know the matrix elements of the coupling Hamiltonian. For this we need to know how to express the charge and flux in terms of a and $a^\dagger$ and hence must know the characteristic impedance of the resonance. Happily, the slope with which the admittance passes through zero determines the characteristic impedance of the resonance

$$\Omega\left(\frac{\partial Y}{\partial\omega}\right)_\Omega = \frac{2j}{Z_0}, \tag{2.174}$$

so that

$$Z_0 = \frac{2j}{\Omega\left(\frac{\partial Y}{\partial\omega}\right)_\Omega}. \tag{2.175}$$

Using Eqs. (2.43-2.44) we can then find any physical quantity we desire.

To see the generality of this result, consider the example of the lumped element circuit in Fig. (2.10). If $L_1 + L_2 = L$ then this has the same bare resonance frequency Ω but clearly will have a different coupling to the port. Use of Eq. (2.44) yields

$$\Phi^2_{\mathrm{ZPF}} = \frac{\hbar\Omega L}{2}\left(\frac{L_1}{L_1 + L_2}\right)^2, \tag{2.176}$$

which is just what we expect from the transformer turns ratio.

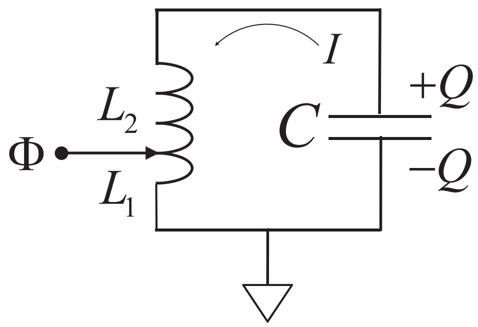

Fig. 2.10 Single port black box containing a simple LC oscillator with the port connected to an inductive divider with $L_1 + L_2 = L$.

Let us suppose for example that we couple to our black box through an inductor L_c as shown in Fig. (2.11). The coupling Hamiltonian is

$$H_1 = \frac{1}{2L_c}\left(\hat{\Phi} - \hat{\Phi}_{\text{in}}\right)^2. \tag{2.177}$$

The operator $\hat{\Phi}_{\text{in}}$ is either a classical control field or is a quantum operator for whatever system we hook up to our black box. Now that we know impedance of the resonance, we know how to express $\hat{\Phi}$ using Eq. (2.44) so that we have

$$H_1 = \frac{1}{2L_c}\left(\Phi_{\text{ZPF}}(\hat{a} + \hat{a}^\dagger) - \hat{\Phi}_{\text{in}}\right)^2. \tag{2.178}$$

The case of capacitive rather than inductive coupling is more complex as can be seen from the example of two capacitively coupled oscillators shown in Fig. (2.6) which we discussed earlier. We found that it was easy to write down the Lagrangian, but finding the Hamiltonian required inverting the capacitance matrix for the entire system. Hence if we are going to use the flux variable at the input port as the coordinate, it is usually easiest to proceed by treating the coupling capacitor as being inside the black box.

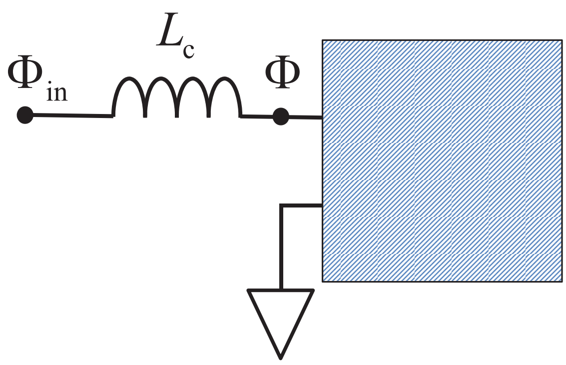

Fig. 2.11 Coupling to a blackbox via an inductor.

The extension of these results to the case of a multi-mode black box Hamiltonian is simply

$$H_0 = \sum_m \Omega_m \hat{a}_m^\dagger \hat{a}_m, \tag{2.179}$$

where the summation is over the different modes and the flux operator at the port of the black box is simply

$$\hat{\Phi} = \sum_m \Phi_{\text{ZPF}}^{(m)}\left(\hat{a}_m + \hat{a}_m^\dagger\right). \tag{2.180}$$

This is simply a statement that the voltage across the input port is the sum of the voltages across each of the resonator elements in series as shown in Fig. (2.9a).

This 'black box' formalism will prove useful if it is possible to either measure, or use finite element simulations to compute, the admittance as a function of frequency. So far we have only discussed quantization of linear circuits which are equivalent to coupled simple harmonic oscillators. Qubits are of course not linear circuit elements, but the formalism developed here is especially useful for the study of transmon qubits coupled to resonators since as we will see in Chap. (4), the transmon qubit is essentially a weakly anharmonic oscillator. The generalization of the discussion above to the coupling of a weakly anharmonic oscillator to a linear black box (Manucharyan *et al.*, 2007) is discussed in detail in Appendix C. The reader should familiarize herself with the discussion of the transmon qubit in Chap. (4) before studying Appendix C.